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1976 J. Phys. A: Math. Gen. 9 269

(http://iopscience.iop.org/0305-4470/9/2/012)

J. Phys. A: Math. Gen., Vol. 9, No. 2, 1976. Printed in Great Britain. © 1976

Further application of the Martin, Siggia, Rose formalism

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Received 16 June 1975, in final form 7 October 1975

Abstract. The functional formalism developed by Martin, Siggia and Rose for classical statistical dynamics and related problems is extended to a wider class of systems. These are characterized by an equation of motion in which a Gaussian random forcing function appears multiplicatively.

1. Introduction

The operator formalism of classical statistical dynamics recently presented by Martin *et al* (1973), to be referred to as the MSR formalism, enables closed functional differential equations to be written for a suitable generating functional which contains complete information about both the correlation and response functions of the system. Such equations provide a very concise formulation of the problem and lead naturally to a consideration of field theory methods in statistical mechanics.

An important feature of the MSR formalism is that it enables one to give a similar description of certain statistical problems of a more general nature in which a random forcing function operates. In a previous paper (Phythian 1975, to be referred to as I) a simple derivation of the MSR formalism has been given for a system with an equation of motion of the form

$$\dot{\psi}_n(t)=f_n(t)+\Lambda_n(\psi(t), t).$$

Here the quantities $\psi_n(t)$ are the dynamical variables of the system, Λ_n are given non-random functions of $\psi(t)$ and t, and the $f_n(t)$ are Gaussian random functions of zero mean. It is convenient to regard n as a discrete index but there is no difficulty in generalizing to a continuous variable. An example of such a system which is of great interest is a fluid in turbulent motion, the system being maintained in a statistically stationary state by random stirring forces.

Although closed-form solutions of some functional differential equations can be obtained in the form of functional integrals, little progress has been made in the evaluation of these and one must resort to more indirect methods. Perhaps the most useful approximation procedure so far developed involves the generation of renormalized series in the way described, for example, by Martin *et al*, and below we shall give a rather simpler version of this procedure. It is possible to obtain such renormalized series by more direct methods but the analysis is very tedious, involving the re-arrangement and partial re-summation of divergent perturbation series with the consequent topological and combinatorial problems. Because of such complexities the first derivations of renormalized series for the turbulence problem were in fact incorrect and a full correct treatment was not published until the paper of Martin *et al* more than ten years later. The correct series had however been previously obtained by Kraichnan in unpublished work using the direct approach. The simplest non-trivial truncation of the renormalized series leads to what has become known in turbulence theory as the direct interaction approximation which has received some experimental support at moderate Reynolds numbers. Higher-order truncations lead to approximations which are very difficult to evaluate but some work has been done on the application of these higher approximations to simpler problems such as an idealized convection problem (Kraichnan 1964) and the randomly forced damped anharmonic oscillator (Morton and Corrsin 1970). This work seems to indicate that the higher approximations give better agreement with the exact results. The same is not apparently true for the turbulence problem itself but this may be due to the unsuitability of a formulation of the problem in Eulerian form.

In addition to providing a simple derivation of renormalized series, the MSR formalism leads naturally to the consideration of other methods first developed for quantum systems. One such method is the variational approach, a preliminary investigation of which has been carried out recently by Rose (1975). Other possibilities are the Edwards expansion procedure and the renormalization group (for references see (I)).

Although it is possible to describe many physical situations by an equation like the one above, there are other cases in which a more general equation is called for. The equation of motion we shall consider here is

$$\dot{\psi}_n(t) = f_n(t) + g_m(t)\Gamma_{mn}(\psi(t), t) + \Lambda_n(\psi(t), t) \tag{1}$$

where a summation over repeated indices is implied, Γ_{mn} and Λ_n are non-random functions of $\psi(t)$ and t, and $f_n(t)$, $g_n(t)$ are independent Gaussian random functions of zero mean with correlation functions given by

$$\langle f_n(t) f_m(t')\rangle = Q_{nm}(t, t')$$
$$\langle g_n(t) g_m(t')\rangle = R_{nm}(t, t').$$

The initial conditions are also random and are statistically independent of both f and g.

We shall first mention briefly some problems which can be so described. Perhaps the simplest of these arises when a passive scalar field $\psi(\boldsymbol{x}, t)$ is advected by a fluid in turbulent motion, the velocity field of which $\boldsymbol{v}(\boldsymbol{x}, t)$ can be approximated by a Gaussian random field. The equation of motion is

$$\frac{\partial\psi}{\partial t} = s(\boldsymbol{x}, t) + \nu\nabla^2\psi - v_\alpha(\boldsymbol{x}, t)\frac{\partial\psi}{\partial x_\alpha}$$

where $s(\boldsymbol{x}, t)$ is a source function which may be non-random or Gaussian, and ν is the molecular diffusivity. By expanding in terms of a complete set of spatial functions the equation is transformed into (1) with Γ and Λ linear functions. A closely related problem is that of the motion of non-interacting particles in a random force field $f(\boldsymbol{x}, t)$, the particle density in phase space $\psi(\boldsymbol{x}, \boldsymbol{v}, t)$ satisfying the equation

$$\frac{\partial\psi}{\partial t} = s(\boldsymbol{x}, \boldsymbol{v}, t) - v_\alpha\frac{\partial\psi}{\partial x_\alpha} - f_\alpha(\boldsymbol{x}, t)\frac{\partial\psi}{\partial v_\alpha}.$$

The corresponding quantum mechanical situation, of interest in the theory of amorphous materials, is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{\hbar^2}{2m}\nabla^2\psi+U(\boldsymbol{x})\psi$$

where $U(\boldsymbol{x})$ is a Gaussian random potential.

The introduction of some form of self-interaction leads to the presence of non-linear terms in the equation of motion. For example the concentration field $\psi(\boldsymbol{x}, t)$ of a substance undergoing an irreversible, isothermal, second-order reaction and being simultaneously advected by a Gaussian velocity field satisfies the equation

$$\frac{\partial\psi}{\partial t}=s(\boldsymbol{x}, t)+\nu\nabla^2\psi-v_\alpha(\boldsymbol{x}, t)\frac{\partial\psi}{\partial x_\alpha}-c\psi^2$$

where c is a constant which determines the reaction rate. Similarly the behaviour of two species A, B undergoing a second-order reaction is given by

$$\frac{\partial\psi_{\mathrm{A}}}{\partial t}=s_{\mathrm{A}}(\boldsymbol{x}, t)+\nu_{\mathrm{A}}\nabla^2\psi_{\mathrm{A}}-v_\alpha(\boldsymbol{x}, t)\frac{\partial\psi_{\mathrm{A}}}{\partial x_\alpha}-c\psi_{\mathrm{A}}\psi_{\mathrm{B}}$$

$$\frac{\partial\psi_{\mathrm{B}}}{\partial t}=s_{\mathrm{B}}(\boldsymbol{x}, t)+\nu_{\mathrm{B}}\nabla^2\psi_{\mathrm{B}}-v_\alpha(\boldsymbol{x}, t)\frac{\partial\psi_{\mathrm{B}}}{\partial x_\alpha}-c\psi_{\mathrm{A}}\psi_{\mathrm{B}}.$$

It is of interest to note that the turbulence problem may also be described by an equation of the form (1). If we define

$$\psi(\boldsymbol{x}, \boldsymbol{v}, t)=\delta(\boldsymbol{v}-\boldsymbol{V}(x, t))$$

where $\boldsymbol{V}$ is the fluid velocity field, then by using the Navier–Stokes equation satisfied by $\boldsymbol{V}$ it can be shown that ψ satisfies the equation

$$\left(\frac{\partial}{\partial t}+v_\alpha\frac{\partial}{\partial x_\alpha}+f_\alpha(\boldsymbol{x}, t)\frac{\partial}{\partial v_\alpha}\right)\psi(\boldsymbol{x}, \boldsymbol{v}, t)$$
$$=\int \mathrm{d}x'\int \mathrm{d}v'\int \mathrm{d}x''\int \mathrm{d}v''\, B(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'')\psi(\boldsymbol{x}', \boldsymbol{v}', t)\psi(\boldsymbol{x}'', \boldsymbol{v}'', t).$$

Here f is the Gaussian stirring force and B is given by

$$B(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{x}', \boldsymbol{v}'; \boldsymbol{x}'', \boldsymbol{v}'')$$
$$=\tfrac{1}{2}\delta(\boldsymbol{x}-\boldsymbol{x}')\left(\frac{\partial}{\partial v_\alpha}\delta(\boldsymbol{v}-\boldsymbol{v}')\right)$$
$$\times\left(\frac{1}{4\pi}v''_\beta v''_\gamma\frac{\partial^3}{\partial x_\alpha\,\partial x_\beta\,\partial x_\gamma}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}''|}-\nu v''_\alpha\nabla^2\,\delta(\boldsymbol{x}-\boldsymbol{x}'')\right)$$
$$+(\text{a similar term with } \boldsymbol{x}', \boldsymbol{v}' \text{ and } \boldsymbol{x}'', \boldsymbol{v}'' \text{ interchanged})$$

where ν is the viscosity. This equation has an advantage over the Navier–Stokes equation in that the non-linear term arises entirely from the true forces (viscous and pressure) acting in the fluid; however this is achieved at the cost of an increase in the number of independent variables. Use of this representation of turbulence has been limited to simple closures of the corresponding hierarchy of equations for the correlation functions of ψ when f is zero (Lundgren 1972). Such approximations involve no renormalization of the response of the system to a perturbing force and are presumably open to the same criticisms as the quasi-normal approximation. It will be apparent later that such a renormalization is more complicated in this representation since it requires the introduction of renormalized vertices. These last examples are all described by equations which can be transformed into (1) by expanding ψ in terms of a suitable complete set of functions. In each case Γ is linear and Λ quadratic in ψ.

It is natural to enquire whether the MSR formalism can accommodate these more general problems, and we shall show that this is indeed the case. One is therefore able to derive renormalized series quite simply and also to apply the other methods mentioned above to a much wider range of systems. We shall here content ourselves with a derivation of the renormalized series. These series do not seem to have appeared before in the literature although the direct interaction approximation, which follows as before from the simplest non-trivial truncation of the series, has been applied to the problem of diffusion in a random velocity field (Kraichnan 1970) where it is found to give good agreement with computer simulations. In this case too it is presumably possible to obtain higher terms of the series by the direct method. The more general case in which Λ is non-linear would be very difficult to deal with by the direct method because of the presence of two stochastic non-linear terms in the equations of motion.

2. Derivation of the functional formalism

The derivation closely follows the treatment given in I and will be described only briefly. The initial values of $\psi_n(t)$ at $t=0$ are denoted by ϕ_n and are specified by a probability density $\rho(\phi)$. We consider functions of ϕ, to be denoted by Greek capital letters Φ, Ψ, etc which form a real Hilbert space with a scalar product given by

$$(\Psi, \Phi) = \int d\phi\, \rho(\phi)\Psi(\phi)\Phi(\phi).$$

By replacing ϕ by $\psi(t)$ in such a function $\Phi(\phi)$ we get a new time-dependent function of ϕ denoted by $\Phi_t(\phi)$. This can be regarded as arising from $\Phi(\phi)$ by the action of a linear evolution operator $E(t)$ so that we have

$$E(t)\Phi(\phi) = \Phi_t(\phi) = \Phi(\psi(t)).$$

Differentiating with respect to t and using the equations of motion (1) we obtain

$$\dot{E}(t) = E(t)\mathscr{L}(t)$$

with $E(0)=1$, where $\mathscr{L}(t)$ is given by

$$\mathscr{L}(t) = (f_n(t) + g_m(t)\Gamma_{mn}(\phi, t) + \Lambda_n(\phi, t))\frac{\partial}{\partial\phi_n}.$$

Similarly

$$\dot{E}^{-1}(t) = -\mathscr{L}(t)E^{-1}(t).$$

Defining operators as follows

$$A_n(t) = E(t)\phi_n E^{-1}(t)$$

$$B_n(t) = E(t)\frac{\partial}{\partial\phi_n}E^{-1}(t)$$

we see that $A_n(t)$ is an operator which simply corresponds to multiplication by $\psi_n(t)$ and is clearly self-adjoint, it will in future be denoted by $\psi_n(t)$. The operator $B_n(t)$ satisfies the equation of motion

$$\dot{B}_n(t) = -g_l(t)\Gamma_{lm,n}(\psi(t), t)B_m(t) - \Lambda_{m,n}(\psi(t), t)B_m(t).$$

It is more convenient to consider the adjoint operator $\hat{\psi}_n(t)$ of $B_n(t)$ which satisfies the equation

$$\dot{\hat{\psi}}_n(t) = -\hat{\psi}_m(t)g_l(t)\Gamma_{lm,n}(\psi(t), t) - \hat{\psi}_m(t)\Lambda_{m,n}(\psi(t), t).$$

We now evaluate the functional derivatives of these operators with respect to the functions f and g. This may be achieved by first evaluating the derivatives of $E(t)$. Writing $X_n(t, t')$ for $\delta E(t)/\delta f_n(t')$ we obtain from the equation of motion for $E(t)$

$$\frac{\partial}{\partial t} X_n(t, t') = X_n(t, t')\mathscr{L}(t) + \delta(t-t')E(t')\frac{\partial}{\partial \phi_n}.$$

Hence

$$\frac{\partial}{\partial t}[X_n(t, t')E^{-1}(t)] = \delta(t-t')B_n(t')$$

and the causal solution for $X_n(t, t')$ is

$$\theta(t-t')B_n(t')E(t).$$

Similarly we have

$$\frac{\delta E(t)}{\delta g_n(t')} = \theta(t-t')\Gamma_{nm}(\psi(t'), t')B_m(t')E(t).$$

Using these results and the definition of $\hat{\psi}_n(t)$ we finally obtain

$$\begin{aligned}
\frac{\delta\psi_n(t)}{\delta f_m(t')} &= \theta(t-t')[\psi_n(t), \hat{\psi}_m(t')] \\
\frac{\delta\hat{\psi}_n(t)}{\delta f_m(t')} &= \theta(t-t')[\hat{\psi}_n(t), \hat{\psi}_m(t')] \\
\frac{\delta\psi_n(t)}{\delta g_m(t')} &= \theta(t-t')[\psi_n(t), \hat{\psi}_l(t')\Gamma_{ml}(\psi(t'), t')] \\
\frac{\delta\hat{\psi}_n(t)}{\delta g_m(t')} &= \theta(t-t')[\hat{\psi}_n(t), \hat{\psi}_l(t')\Gamma_{ml}(\psi(t'), t')].
\end{aligned} \tag{2}$$

These expressions for functional derivatives in terms of causal commutators play an essential role in the derivation of the functional formalism since they allow one to write the response functions in terms of time-ordered products of the operators ψ, $\hat{\psi}$. For example we have

$$\mathscr{E}\left\{\frac{\delta\psi_n(t)}{\delta g_m(t')}\right\} = \langle(\Phi_0, T\{\psi_n(t)\hat{\psi}_l(t'+)\Gamma_{ml}(\psi(t'), t')\}\Phi_0)\rangle.$$

Here Φ_0 is the unit function, T the time-ordering operator, the angular brackets denote the expectation value over the random functions f and g, and $\mathscr{E}$ includes the additional average over initial data. The correlation functions can be written in the same notation, for example

$$\mathscr{E}\{\psi_n(t)\psi_m(t')\} = \langle(\Phi_0, T\{\psi_n(t)\psi_m(t')\}\Phi_0)\rangle.$$

Such expectation values can be expressed in terms of the generating functional

$$Z = \left\langle\left(\Phi_0, T\exp\int \mathrm{d}t\,(\xi_n(t)\psi_n(t) + \eta_n(t)\hat{\psi}_n(t))\Phi_0\right)\right\rangle.$$

The functional differential equations for Z are obtained as in (1) by using the equations of motion for the operators ψ, $\hat{\psi}$ together with Novikov's theorem and the expressions (2) for functional derivatives. We find

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\delta Z}{\delta\xi_n(t)} &= \eta_n(t)Z + \Lambda_n\left(\frac{\delta}{\delta\xi(t)}, t\right)Z + \int \mathrm{d}t'\, Q_{nm}(t, t')\frac{\delta Z}{\delta\eta_m(t')} \\
&\quad + \int \mathrm{d}t'\, R_{mp}(t, t')\frac{\delta}{\delta\eta_q(t'+)}\Gamma_{pq}\left(\frac{\delta}{\delta\xi(t')}, t'\right)\Gamma_{mn}\left(\frac{\delta}{\delta\xi(t)}, t\right)Z \\
\frac{\partial}{\partial t}\frac{\delta Z}{\delta\eta_n(t)} &= -\xi_n(t)Z - \frac{\delta}{\delta\eta_m(t+)}\Lambda_{m,n}\left(\frac{\delta}{\delta\xi(t)}, t\right)Z \\
&\quad - \int \mathrm{d}t'\, R_{pq}(t, t')\frac{\delta}{\delta\eta_r(t'+)}\Gamma_{qr}\left(\frac{\delta}{\delta\xi(t')}, t'\right)\frac{\delta}{\delta\eta_m(t+)}\Gamma_{pm,n}\left(\frac{\delta}{\delta\xi(t)}, t\right)Z.
\end{aligned} \tag{3}$$

The equations may easily be rewritten in coordinate representation. For example for the first problem described we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\frac{\delta Z}{\delta\xi(\boldsymbol{x}, t)} \\
&\qquad = \eta(\boldsymbol{x}, t)Z + \int \mathrm{d}\boldsymbol{x}'\int \mathrm{d}t'\, Q(\boldsymbol{x}, t; \boldsymbol{x}', t')\frac{\delta Z}{\delta\eta(\boldsymbol{x}', t')} \\
&\qquad\quad + \int \mathrm{d}\boldsymbol{x}'\int \mathrm{d}t' R_{\alpha\beta}(\boldsymbol{x}, t; \boldsymbol{x}', t')\frac{\delta}{\delta\eta(\boldsymbol{x}', t+)}\frac{\partial}{\partial x'_\beta}\frac{\delta}{\delta\xi(\boldsymbol{x}', t')}\frac{\partial}{\partial x_\alpha}\frac{\delta Z}{\delta\xi(\boldsymbol{x}, t)} \\
&\left(\frac{\partial}{\partial t} + \nu\nabla^2\right)\frac{\delta Z}{\delta\eta(\boldsymbol{x}, t)} \\
&\qquad = -\xi(\boldsymbol{x}, t)Z \\
&\qquad\quad + \int \mathrm{d}\boldsymbol{x}'\int \mathrm{d}t'\, R_{\alpha\beta}(\boldsymbol{x}, t; \boldsymbol{x}', t')\frac{\delta}{\delta\eta(\boldsymbol{x}', t'+)}\frac{\partial}{\partial x'_\beta}\frac{\delta}{\delta\xi(\boldsymbol{x}', t')}\frac{\partial}{\partial x_\alpha}\frac{\delta Z}{\delta\eta(\boldsymbol{x}, t)}
\end{aligned} \tag{4}$$

where

$$\langle s(\boldsymbol{x}, t)s(\boldsymbol{x}', t')\rangle = Q(\boldsymbol{x}, t; \boldsymbol{x}', t')$$
$$\langle v_\alpha(\boldsymbol{x}, t)v_\beta(\boldsymbol{x}', t')\rangle = R_{\alpha\beta}(\boldsymbol{x}, t; \boldsymbol{x}', t').$$

It is seen that when the correlation functions Q and R contain delta functions in the time differences, one may write down equivalent equations of motion for the operators ψ, $\hat{\psi}$ in which the random functions f, g do not appear. This is a generalization of an observation made by Martin *et al.*

3 Derivation of renormalized expansions

As mentioned earlier the functional differential equations provide the simplest derivation of the renormalized expansions for correlation and response functions. To illustrate this point let us consider first the simple case of the statistically stationary state arising from the equation

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\psi(\boldsymbol{x}, t)=s(\boldsymbol{x}, t)-v_\alpha(\boldsymbol{x}, t)\frac{\partial\psi(\boldsymbol{x}t)}{\partial \boldsymbol{x}_\alpha}$$

with the source term s identically zero. Although the stationary state in the absence of sources is clearly one in which ψ is zero so that all correlation functions are zero, the response functions are non-zero. For example the quantity $G(x, t; x', t')=\delta\psi(\boldsymbol{x}, t)/\delta s(\boldsymbol{x}', t')$ satisfies the equation

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2+v_\alpha\frac{\partial}{\partial x_\alpha}\right)G(\boldsymbol{x}, t; \boldsymbol{x}', t')=\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t').$$

This is a case of interest since for $\nu=0$ it corresponds to the motion of marked particles in turbulent flow. The expectation value of $G(\boldsymbol{x}, t; \boldsymbol{x}', t')$ gives the probability that a particle which was at $\boldsymbol{x}'$ at time t' will be found at $\boldsymbol{x}$ at time t.

The corresponding functional differential equations are given by (4) with Q zero. If we integrate the first of these forward in time from $t=0$, using the fact that the initial conditions are eventually forgotten as the stationary state is approached, and integrate the second backward in time from $t=\infty$ using the fact that $\delta Z/\delta\eta(\boldsymbol{x}, t)\to 0$ as $t\to\infty$, since the corresponding time-ordered operator product has a $\hat{\psi}$ on the left, we can rewrite the equations in the form

$$\frac{\delta Z}{\delta\xi(1)}=G_0(1, 2)\eta(2)Z+G_0(1, 2)C(2, 3, 4, 5)\frac{\delta^3 Z}{\delta\eta(3)\,\delta\xi(4)\,\delta\xi(5)}$$

$$\frac{\delta Z}{\delta\eta(1)}=G_0(2, 1)\xi(2)Z+G_0(2, 1)C(5, 4, 3, 2)\frac{\delta^3 Z}{\delta\eta(5)\,\delta\eta(4)\,\delta\xi(3)}.$$

Here we have used the usual shorthand notation in which 1 stands for $\boldsymbol{x}_1$, t_1 and integration over repeated variables is implied. $G_0(1, 2)$ is the Green function

$$\theta(t_1-t_2)[4\pi\nu(t_1-t_2)]^{-3/2}\exp\left(-\frac{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2}{4\nu(t_1-t_2)}\right)$$

and C is given by

$$C(1, 2, 3, 4)=\frac{1}{2}\left[R_{\alpha\beta}(\boldsymbol{x}_1, t_1; \boldsymbol{x}_2, t_2)\left(\frac{\partial}{\partial x_{1\alpha}}\delta(\boldsymbol{x}_1-\boldsymbol{x}_4)\right)\left(\frac{\partial}{\partial x_{2\beta}}\delta(\boldsymbol{x}_2-\boldsymbol{x}_3)\right)\right.$$
$$\left.\times\,\delta(t_1-t_4-)\,\delta(t_2-t_3-)\right]+(\text{similar term with 3 and 4 interchanged}).$$

It is clear that C has the symmetry property

$$C(1, 2, 3, 4)=C(2, 1, 3, 4)=C(1, 2, 4, 3)=C(2, 1, 4, 3).$$

Following Martin *et al* we now further simplify the equations by introducing an extra 'spin' variable σ which can have only the two values ± 1, and define

$$\zeta(1,+)=\xi(1), \qquad \zeta(1,-)=\eta(1)$$
$$\mathscr{G}_0(1,+;2,-)=G_0(1,2)$$
$$\mathscr{G}_0(1,-;2,+)=G_0(2,1)$$
$$K(1,-;2,-;3,+;4,+)=\tfrac{1}{3}C(1,2,3,4)$$

with K symmetric with respect to all permutations of the variables including spin, i.e.

$$K(1,-;2,-;3,+;4,+)=K(3,+;2,-;1,-;4,+)=\text{etc.}$$

We shall henceforth absorb the spin variables so that 1 now stands for x_1, t_1, σ_1. The equation finally takes the form

$$\frac{\delta Z}{\delta\zeta(1)}=\mathscr{G}_0(1,2)\zeta(2)Z+\mathscr{G}_0(1,2)K(2,3,4,5)\frac{\delta^3 Z}{\delta\zeta(3)\,\delta\zeta(4)\,\delta\zeta(5)}.$$

Writing $Z=\exp W$ we get the equation for W

$$\frac{\delta W}{\delta\zeta(1)}=\mathscr{G}_0(1,2)\zeta(2)+\mathscr{G}_0(1,2)K(2,3,4,5)\left(\frac{\delta^3 W}{\delta\zeta(3)\,\delta\zeta(4)\,\delta\zeta(5)}+3\frac{\delta W}{\delta\zeta(3)}\frac{\delta^2 W}{\delta\zeta(4)\,\delta\zeta(5)}+\frac{\delta W}{\delta\zeta(3)}\frac{\delta W}{\delta\zeta(4)}\frac{\delta W}{\delta\zeta(5)}\right). \tag{5}$$

Since we are dealing with a situation in which the mean value of ψ is zero it is clear that W contains no term linear in ζ. The perturbation series for W is obtained by writing down a solution as a 'power series' in K. The renormalized expansion is obtained by considering the equation

$$\begin{aligned}\frac{\delta W}{\delta\zeta(1)}&=\mathscr{G}(1,2)\zeta(2)+\mathscr{G}(1,2)[\lambda\Sigma_1(2,3)+\lambda^2\Sigma_2(2,3)+\cdots]\frac{\delta W}{\delta\zeta(3)}\\&\quad+\lambda\mathscr{G}(1,2)[M(2,3,4,5)+\lambda M_1(2,3,4,5)+\lambda^2 M(2,3,4,5)+\ldots]\\&\quad\times\left[\frac{\delta^3 W}{\delta\zeta(3)\,\delta\zeta(4)\,\delta\zeta(5)}+3\frac{\delta W}{\delta\zeta(3)}\frac{\delta^2 W}{\delta\zeta(4)\,\delta\zeta(5)}+\frac{\delta W}{\delta\zeta(3)}\frac{\delta W}{\delta\zeta(4)}\frac{\delta W}{\delta\zeta(5)}\right]\end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathscr{G}_0^{-1}&=\mathscr{G}^{-1}-\Sigma_1-\Sigma_2-\ldots\\K&=M+M_1+M_2+\ldots\end{aligned} \tag{7}$$

and Σ_n, M_n have the same symmetry as $\mathscr{G}_0$ and K. It is apparent that for $\lambda=1$ the equation reduces to the one to be solved, while for $\lambda=0$ it reduces to

$$\frac{\delta W}{\delta\zeta(1)}=\mathscr{G}(1,2)\zeta(2)$$

which has the solution

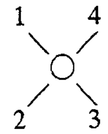
$$W_0 = \tfrac{1}{2}\zeta(1)\mathscr{G}(1,2)\zeta(2).$$

We seek a solution of (6) as a power series in λ,

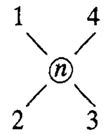
$$W_0 + \lambda W_1 + \lambda^2 W_2 + \ldots .$$

The terms of the series are most conveniently represented in diagram form as follows:
$\mathscr{G}(1,2)$ is represented by a line joining the points 1, 2, ^{1}——^{2}
$\zeta(1)$ by a small black circle at the point 1, •

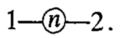
$M(1,2,3,4)$ by



$M_n(1,2,3,4)$ by



$\Sigma_n(1,2)$ by



We can therefore write

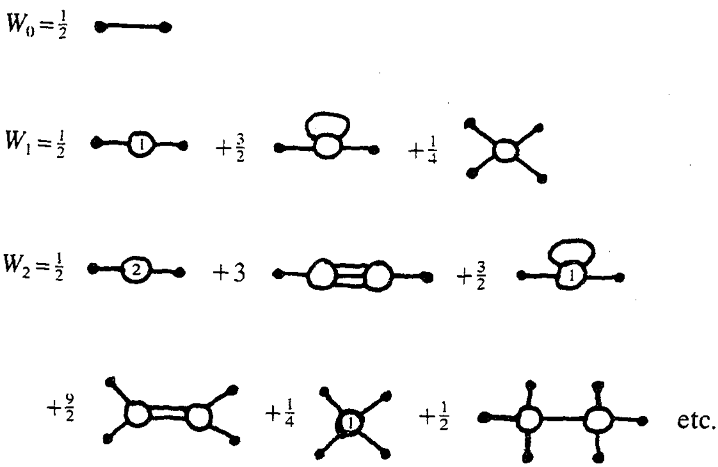


Figure 1.

We now impose the condition that the only terms in the series which are quadratic in ζ are those in W_0 and the only quartic terms are those in W_1. There are thus no contributions to $(\delta^2 W/\delta\zeta^2)_0$ and $(\delta^4 W/\delta\zeta^4)_0$ from higher-order terms. These conditions determine the quantities Σ_n, M_n uniquely in terms of $\mathscr{G}$ and M. We obtain

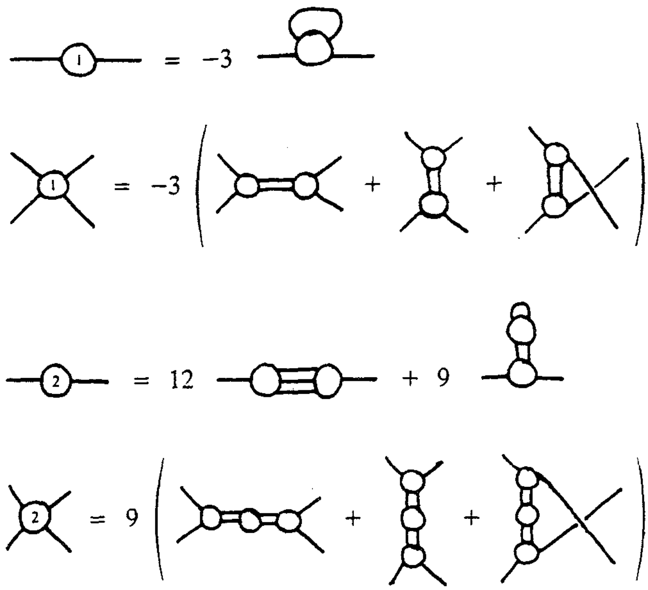


Figure 2.

It should be borne in mind that the present notation is a highly condensed one so that M actually represents three different sorts of vertex in the more usual notation (four if sources were present). If these expressions are now substituted into (7) we obtain the desired renormalized series. The direct interaction approximation is obtained by neglecting all terms except Σ_1 in these equations, as may easily be verified.

The generalization to the case when there is also a quadratic term in the equations of motion is not difficult. Using the same notation, the equation for W, the logarithm of the generating functional, is

$$\frac{\delta W}{\delta\zeta(1)} = \mathscr{G}_0(1,2)\zeta(2) + \mathscr{G}_0(1,2)J(2,3,4)\left(\frac{\delta^2 W}{\delta\zeta(3)\,\delta\zeta(4)} + \frac{\delta W}{\delta\zeta(3)}\frac{\delta W}{\delta\zeta(4)}\right)$$
$$+\mathscr{G}_0(1,2)K(2,3,4,5)$$
$$\times\left(\frac{\delta^3 W}{\delta\zeta(3)\,\delta\zeta(4)\,\delta\zeta(5)} + 3\frac{\delta W}{\delta\zeta(3)}\frac{\delta^2 W}{\delta\zeta(4)\,\delta\zeta(5)} + \frac{\delta W}{\delta\zeta(3)}\frac{\delta W}{\delta\zeta(4)}\frac{\delta W}{\delta\zeta(5)}\right)$$

where the exact form of $\mathscr{G}_0$, J, K depends on the particular problem considered. In general the mean value of the field will be non-zero so that W contains a term linear in ζ. It is convenient to work in terms of a functional which contains no linear term, which is equivalent to using dynamical variables of zero mean. We therefore write

$$W = \beta(1)\zeta(1) + \bar{W}$$

where

$$\beta(1) = \left(\frac{\delta W}{\delta\zeta(1)}\right)_0.$$

The quantity $\bar{W}$ satisfies the equation

$$\frac{\delta\bar{W}}{\delta\zeta(1)} = \bar{\mathscr{G}}_0(1,2)\zeta(2) + \bar{\mathscr{G}}_0(1,2)\gamma(2) + \bar{\mathscr{G}}_0(1,2)\bar{J}(2,3,4)\left(\frac{\delta^2\bar{W}}{\delta\zeta(3)\,\delta\zeta(4)} + \frac{\delta\bar{W}}{\delta\zeta(3)}\frac{\delta\bar{W}}{\delta\zeta(4)}\right)$$
$$+\bar{\mathscr{G}}_0(1,2)K(2,3,4,5)\left(\frac{\delta^3\bar{W}}{\delta\zeta(3)\,\delta\zeta(4)\,\delta\zeta(5)} + 3\frac{\delta\bar{W}}{\delta\zeta(3)}\frac{\delta^2\bar{W}}{\delta\zeta(4)\,\delta\zeta(5)}\right.$$
$$\left.+\frac{\delta\bar{W}}{\delta\zeta(3)}\frac{\delta\bar{W}}{\delta\zeta(4)}\frac{\delta\bar{W}}{\delta\zeta(5)}\right)$$

where $\bar{\mathscr{G}}_0, \gamma, \bar{J}$ are related to $\mathscr{G}_0, J, K, \beta$ and the functions $(\delta^2 W/\delta\zeta^2)_0$ and $(\delta^3 W/\delta\zeta^3)_0$ by equations which we do not give here.

The renormalized expansion is obtained from the equation

$$\mathscr{G}^{-1}(1,2)\frac{\delta\bar{W}}{\delta\zeta(2)}$$
$$= \zeta(1) + \sum\lambda^n\mu^m\alpha_{nm}(1) + \sum\lambda^n\mu^m\Sigma_{nm}(1,2)\frac{\delta\bar{W}}{\delta\zeta(2)}$$
$$+\lambda\sum\lambda^n\mu^m Y_{nm}(1,2,3)\left(\frac{\delta^2\bar{W}}{\delta\zeta(2)\,\delta\zeta(3)} + \frac{\delta\bar{W}}{\delta\zeta(2)}\frac{\delta\bar{W}}{\delta\zeta(3)}\right)$$
$$+\mu\sum\lambda^n\mu^m X_{nm}(1,2,3,4)\left(\frac{\delta^3\bar{W}}{\delta\zeta(2)\,\delta\zeta(3)\,\delta\zeta(4)}\right.$$
$$\left.+3\frac{\delta\bar{W}}{\delta\zeta(2)}\frac{\delta^2\bar{W}}{\delta\zeta(3)\,\delta\zeta(4)} + \frac{\delta\bar{W}}{\delta\zeta(2)}\frac{\delta\bar{W}}{\delta\zeta(3)}\frac{\delta\bar{W}}{\delta\zeta(4)}\right)$$

where

$$\alpha_{00} = \Sigma_{00} = 0$$
$$\mathscr{G}^{-1} - \sum\Sigma_{nm} = \bar{\mathscr{G}}_0^{-1}$$
$$\sum Y_{nm} = \bar{J}$$
$$\sum X_{nm} = K$$
$$\sum\alpha_{nm} = \gamma.$$

For $\lambda = \mu = 1$ the equation reduces to the one of interest, and for $\lambda = \mu = 0$ it reduces to an equation with solution

$$W_{00} = \tfrac{1}{2}\zeta(1)\mathscr{G}(1,2)\zeta(2).$$

We expand the solution about W_{00} as a double power series and then determine the quantities $\alpha_{nm}, \Sigma_{nm}, Y_{nm}, X_{nm}$ from the requirement that terms linear in ζ vanish while terms quadratic, cubic, and quartic in ζ occur only in the lowest possible orders in the expansion parameters λ, μ. The renormalized series are then found as before and we obtain

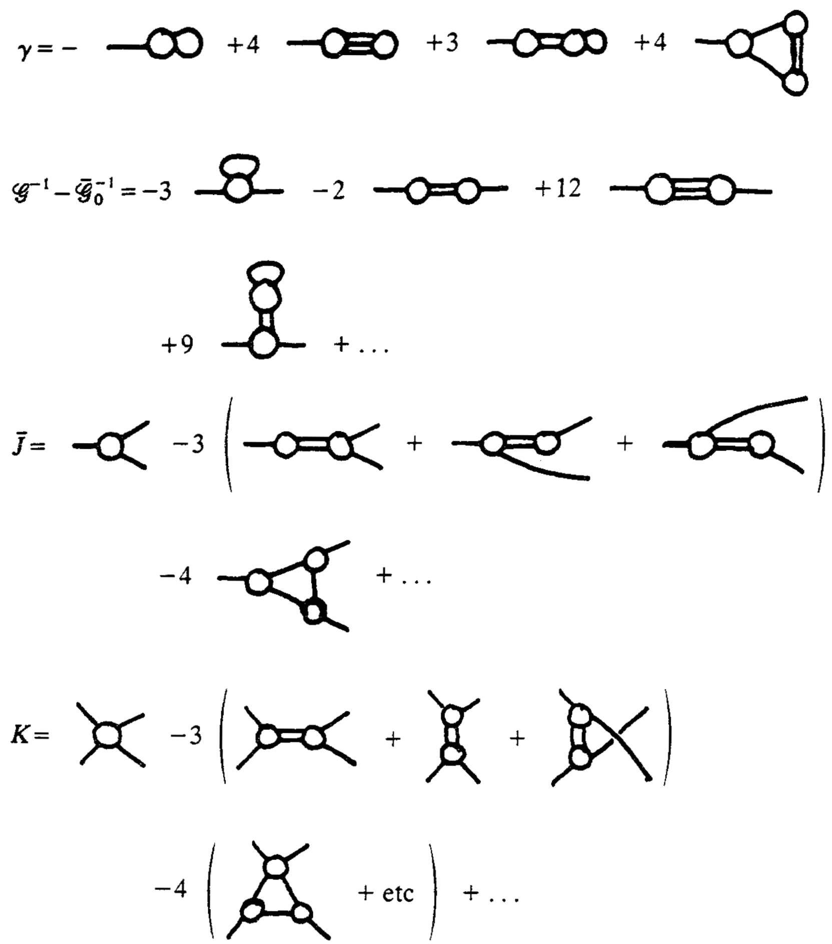


Figure 3.

Acknowledgment

The author is grateful to Dr H A Rose for sending a copy of his paper prior to publication.

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